A Note on Robust Multifractal Spectra

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Abstract

Here we delineate a robust approach for estimating the wavelet-based multifractal spectrum (MFS) in two dimensions considered by Ramírez-Cobo and Vidakovic (2012).

Key words: Multiscale analysis of 2D data; 2D-wavelet transform; Robust statistics; Mammograms images; Fractional Brownian motion.

1 Introduction

Fractal and multifractal approaches have been used to analyze a variety of signals in areas such as finance, geophysics, web, teletraffic and medicine, see Chhabra et al. (1989), Mandelbrot (1989), Mandelbrot (1997), Riedi (1999) and Reljin (2002). These signals are characterized by the implicit occurrence of irregularities and certain degree of self-similarity over a range of scales. One the tools considered in the literature for analyzing self-similar signals is the multi fractal spectrum (Riedi, 1999), which quantifies different degrees of scaling existing in a signal.

Because of the number of application in medicine, climatology and geophysics (see Netsch 1999, Shi 2005), among others, in this paper we focus on two-dimensional data. Ramírez-Cobo and Vidakovic (2012) considered the extension of the multi fractal spectrum to the two-dimensional case using wavelet transforms. In that work the authors also suggest an estimation

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method along the lines of Goncalves (1998), and applied their approach for the analysis of digitized mammograms. The purpose of this work is to consider robust estimation of the 2D multi fractal spectrum, where the motivation is to take into account the possible violations in the model assumptions presented by real-life datasets (frequency-dependent noise, non-Gaussianity, outlier multiresolution levels). Our approach is based on Theil-type weighted regression (Theil, 1950), where average multiresolution level "energies" are regressed against the level indices and builds on work of Hamilton et al (2011).

2 Preliminaries

2.1 Background on wavelets

The discrete wavelet transform expresses a real signal $X(t)$ in terms of shifted and dilated versions of a wavelet (or *mother*) function $\psi(t)$ and shifted versions of a scaling (or *father*) function $\phi(t)$. For specific choices of the scaling functions and wavelets, an orthonormal basis can be formed from the atoms

$$\begin{aligned}\psi_{j,k}(t) &= 2^{j/2}\psi(2^j t - k)\\ \phi_{j,k}(t) &= 2^{j/2}\phi(2^j t - k), \quad j,k \in \mathbb{Z}.\end{aligned}$$

The signal $X(t)$ can be thus represented by wavelets as

$$X(t) = \sum_k c_{J_0,k}\phi_{J_0,k}(t) + \sum_{j=J_0}^{\infty}\sum_k d_{j,k}\psi_{j,k}(t),$$

where

$$d_{j,k} = \int X(t)\psi_{j,k}(t)dt \quad \text{and} \tag{1}$$

$$c_{j,k} = \int X(t)\phi_{j,k}(t)dt, \tag{2}$$

are detail and scaling coefficients. Here, J_0 indicates the coarsest scale or lowest resolution of analysis, and a larger j corresponds to higher resolutions. For a detailed wavelets theory, we refer to the reader to Daubechies (1992) or Mallat (1997). In practice, many signals are multidimensional. Examples include measurements in geophysics, medicine, astronomy, economics, and so on. The wavelet transform is readily generalized to the multidimensional case. Since we are interested in the wavelet transforms of images, the generalization we show is for the two-dimensional case. The 2D wavelet bases functions are constructed via translations and dilations of a tensor product of univariate wavelets and scaling functions:

$$\begin{aligned}\phi(t_1, t_2) &= \phi(t_1)\phi(t_2)\\ \psi^h(t_1, t_2) &= \phi(t_1)\psi(t_2)\\ \psi^v(t_1, t_2) &= \psi(t_1)\phi(t_2)\\ \psi^d(t_1, t_2) &= \psi(t_1)\psi(t_2).\end{aligned} \tag{3}$$

The symbols h, v, d in (3) stand for horizontal, vertical and diagonal directions, respectively. Consider the wavelet atoms

$$\begin{aligned}\phi_{j,\mathbf{k}}(\mathbf{t}) &= 2^{2j}\phi(2^j t_1 - k_1, 2^j t_2 - k_2)\\ \psi^i_{j,\mathbf{k}}(\mathbf{t}) &= 2^{2j}\psi^i(2^j t_1 - k_1, 2^j t_2 - k_2),\end{aligned}$$

for $i = h, v, d$ and where $\mathbf{t} = (t_1, t_2) \in \mathbb{R}^2$, and $\mathbf{k} = (k_1, k_2) \in \mathbb{Z}^2$. Then, any function $X \in \mathbb{L}^2(\mathbb{R}^2)$ can be represented as

$$X(\mathbf{t}) = \sum_{\mathbf{k}} c_{J_0\mathbf{k}}\phi_{J_0,\mathbf{k}}(\mathbf{t}) + \sum_{j \geqslant J_0}\sum_{\mathbf{k}}\sum_{i} d^i_{j,\mathbf{k}}\psi^i_{j,\mathbf{k}}(\mathbf{t}), \tag{4}$$

where the wavelet coefficients are given by

$$d^i_{j,\mathbf{k}} = 2^{2j}\int X(\mathbf{t})\psi^i(2^j\mathbf{t} - \mathbf{k})d\mathbf{t}. \tag{5}$$

2.2 A 2D Wavelet-based Multifractal Spectrum

Consider a two-dimensional process or signal X that, at a given time $\mathbf{t}$, is assumed to be Hölder continuous,

$$|X(\mathbf{s}) - X(\mathbf{t})| = \mathcal{O}(||\mathbf{s} - \mathbf{t}||^\alpha), \quad \text{for} \quad \mathbf{s} \to \mathbf{t},$$

where $\alpha > 0$ and $||\cdot||$ is the usual Euclidean norm in $\mathbb{R}^2$. Assume that X admits a representation given by (4). Then, Lemma 2.7 in Riedi (1999) generalizes to show that

$$|d^i_{j,\mathbf{k}}| = \mathcal{O}(2^{-j\alpha}), \quad j \to \infty, \tag{6}$$

where the coefficients $d^i_{j,\mathbf{k}}$ were defined in (5). This suggests that the elements $d^i_{j,\mathbf{k}}$ can describe the local oscillatory behavior of X. A *coarse wavelet singularity exponents* of X can be defined as

$$w^i_{j,\mathbf{k}} := -\frac{1}{j}\log_2|d^i_{j,\mathbf{k}}|. \tag{7}$$

Following Riedi (1999) a wavelet-based *local singularity exponent* can be obtained from (7)

$$\alpha^i(\mathbf{t}) := \liminf_{\mathbf{k}2^{-\mathrm{j}} \to \mathbf{t}} w^i_{j,\mathbf{k}}, \tag{8}$$

where $\mathbf{k}2^{-j} \to \mathbf{t}$ means that $\mathbf{t} = (t_1, t_2) \in [2^{-j}k_1, 2^{-j}(k_1+1)] \times [2^{-j}k_2, 2^{-j}(k_2+1)]$, for $\mathbf{k} = (k_1, k_2)$ and $j \to \infty$. The index i in (8) corresponds to one of three directions in detail spaces of 2D wavelet transform: horizontal (h), vertical (v) or diagonal (d). Smaller values of $\alpha^i(\mathbf{t})$ correspond to larger oscillations in X and thus to more singularity (or irregularity) at time $\mathbf{t}$. Typically, a process will possess many different singularity strengths. The frequency (in $\mathbf{t}$) of occurrence of a given singularity strength α is measured by the 2D-MFS, defined for each direction $i = d, h, v$ as

$$f^i(\alpha) := \lim_{\epsilon \to 0} \liminf_{j \to \infty} \frac{1}{j} \log_2 N^i_j(\alpha, \epsilon), \tag{9}$$

where

$$N^i_j(\alpha, \epsilon) := \# \{\mathbf{k} : \ \alpha - \epsilon \leq w^i_{j,\mathbf{k}} < \alpha + \epsilon\}, \tag{10}$$

for $\mathbf{k} \in \{0, \ldots, 2^j - 1\} \times \{0, \ldots, 2^j - 1\}$. The 2D-MFS f^i defined as in (9) is hard to calculate. A practical approach makes use of the theory of large deviations (Ellis, 1984), where f^i would be interpreted as the rate function of a Large Deviation Principle (Riedi, 1999). For a fixed realization of X, $N^i_j(\alpha, \epsilon)/2^{2j}$ can be considered as the probability to find a value $\mathbf{k}$ $\in \{0, \ldots, 2^j - 1\} \times \{0, \ldots, 2^j - 1\}$ such that $w^i_{j,\mathbf{k}} \in [\alpha - \epsilon, \alpha + \epsilon]$. Typically, there will be a most frequent value of $\alpha^i(\mathbf{t})$, denoted by H, and $f^i(\alpha)$ will reach its maximum at $\alpha = H$. On the other hand, if α is different from H, then $[\alpha - \epsilon, \alpha + \epsilon]$ will not contain H for small ϵ and the chance to observe exponents $w^i_{j,\mathbf{k}}$ which lie in $[\alpha - \epsilon, \alpha + \epsilon]$ will decrease exponentially fast with rate given by $f^i(\alpha)$. In this context, the scaling behavior of the moments of the wavelets coefficients (5) is studied. For every direction i, the *partition* or *moment scaling* function is defined,

$$\tau^i(q) := \liminf_{j \to \infty} -\frac{1}{j} \log_2 \mathbb{E}|d^i_{j,\mathbf{k}}|^q. \tag{11}$$

The partition function (11) describes the limiting behaviour of qth moment of a typical wavelet coefficient from the level j and direction i. Under some technical conditions, the *multifractal formalism* (Riedi, 1999; Riedi et al. 1999) posits that the multifractal spectrum can be calculated via the Legendre transform

$$f^i(\alpha) = f^i_L(\alpha) := \inf_q [q\alpha - \tau^i(q)].$$

It can be shown that $f^i_L(\alpha) = q\alpha - \tau^i(q)$ at $\alpha = \tau'^i(q)$ provided $\tau''^i(q) < 0$.

3 The MFS of a 2D Fractional Brownian Motion

The fractional Brownian motion (fBm) is arguably the most popular statistical model in signal and image processing for description of data that scale in a regular fashion. The fBm has proved useful for modeling various physical phenomena involving long-range dependence and regular self-similarity. It is a natural extension (Mandelbrot and van Ness, 1968) of the well-known Brownian motion and can be defined as the unique Gaussian, zero-mean process, $B_H(t)$, which is self-similar and has stationary increments.

The definition of the fractional Brownian motion can be extended to higher dimensions along the lines of (Lévy, 1948), where the generalization of Brownian motion to multiple dimensions was first considered. A 2D-fBm, $B_H(\mathbf{t})$, for $\mathbf{t} \in [0,1] \times [0,1]$ and $H \in (0,1)$, is a process with stationary zero-mean Gaussian increments, for which

$$B_H(a\mathbf{t}) \stackrel{d}{=} a^H B_H(\mathbf{t}), \tag{12}$$

and where the autocovariance function is given by

$$E\left(B_H(\mathbf{t})B_H(\mathbf{s})\right) = \frac{\sigma^2}{2}\left(\|\mathbf{t}\|^{2H} + \|\mathbf{s}\|^{2H} - \|\mathbf{t}-\mathbf{s}\|^{2H}\right),$$

where $\|\cdot\|$ is the usual Euclidean norm in $\mathbb{R}^2$. The index H corresponds to the Hurst exponent; a higher exponent H corresponds to a more regular fBm surfaces.

Wavelet coefficients of a 2D-fBm are given by

$$d_{j,\mathbf{k}} = 2^{2j}\int B_H(\mathbf{t})\psi(2^j\mathbf{t}-\mathbf{k})\mathrm{d}\mathbf{t},$$

where the integral is taken over $\mathbb{R}^2$, $\mathbf{k} = (k_1, k_2) \in \mathbb{Z}^2$, and direction $i = d, v$ or h. Note that $d_{j,\mathbf{k}}$ is L^1-normalized which simplifies the property (15) below. With a change of variable in the previous integral, taking into account the self-similarity (12) and following Flandrin (1992), it can be shown the wavelet coefficients have the following properties:

$$d_{j,\mathbf{k}} \stackrel{d}{=} d_{j,0}, \quad \forall\mathbf{k} \tag{13}$$

$$d_{j,\mathbf{k}} \stackrel{d}{=} 2^{-jH}d_{0,\mathbf{k}}, \tag{14}$$

$$d_{j,\mathbf{k}} \sim N(0, 2^{-2jH}\sigma_\psi), \tag{15}$$

for all $\mathbf{k}$, j and where σ_ψ is a constant depending only on the wavelet function ψ. Properties (13)-(15) are called isotropy, scaling and Gaussianity, respectively. It can be seen that the partition function (11) for the 2D-fBm is given by

$$\tau(q) = qH, \quad q > -1, \tag{16}$$

where for simplicity, the index i has been dropped.

Next, we describe how statistical estimation of the 2D-fBm may be carried out. Given a realization of the 2D-fBm of size $2^J \times 2^J$, and using the stationarity of the wavelets coefficients $\{d_{j,(k_1,k_2)}; j = J_0, \ldots, J-1, k_1, k_2 = 0, \ldots, 2^j - 1\}$, the sample counterpart of $\mathbb{E}|d_{j,\mathbf{k}}|^q$ is

$$\begin{aligned} \overline{|d_{j,(k_1,k_2)}|^q} &= \frac{1}{2^j} \sum_{k_1=0}^{2^j-1} \left(\frac{1}{2^j} \sum_{k_2=0}^{2^j-1} |d_{j,(k_1,k_2)}|^q \right) \\ &\stackrel{d}{=} 2^{-jHq}\, \overline{|d_{0,(k_1,k_2)}|^q}. \end{aligned} \tag{17}$$

From (11),

$$2^{-j\tau(q)} \sim \mathbb{E}|d_{j,\mathbf{k}}|^q,$$

and thus, the partition function (11) can be estimated as the power-law exponent of the variation of $\overline{|d_{j,(k_1,k_2)}|^q}$ versus scale 2^{-j}. By weighted linear regression of $\log_2 \overline{|d_{j,(k_1,k_2)}|^q}$ on j between scales j_1 and j_2 we get

$$\widehat{\tau}(q) := \sum_{j=j_1}^{j_2} a_j \log_2 \overline{|d_{j,(k_1,k_2)}|^q},$$

where the regression weights a_j must verify Abry et al. (1999) the two conditions: $\sum_j a_j = 0$ and $\sum_j j a_j = 1$. Thus, we can estimate $f(\alpha)$ though a local slope of $\widehat{\tau}(q)$ at values

$$\widehat{\alpha}(q_l) = [\widehat{\tau}(q_{l+1}) - \widehat{\tau}(q_l)]/q_0, \quad q_l = l q_0$$

as

$$\widehat{f}(\alpha(q_l)) = q_l \alpha(q_l) - \widehat{\tau}(q_l). \tag{18}$$

4 The Robust MF Spectra

The robust estimation approach builds on results of Hamilton et al (2011).

From (15), $d_{0,(k_1,k_2)}$ is a random variable normally distributed with $Ed_{0,(k_1,k_2)} = 0$ and $Vd_{0,(k_1,k_2)} = \sigma_\psi$, then for $q > -1$,

$$E|d_{0,(k_1,k_2)}|^q = \sigma_\psi^q \ \frac{2^{q/2}\Gamma\left(\frac{1+q}{2}\right)}{\sqrt{\pi}},$$

from which

$$V|d_{0,(k_1,k_2)}|^q = \sigma_\psi^{2q} \ 2^q \left(\frac{\Gamma\left(\frac{1+2q}{2}\right)}{\sqrt{\pi}} - \frac{\Gamma^2\left(\frac{1+q}{2}\right)}{\pi} \right)$$

is obtained. From the CLT,

$$\overline{|d_{0,(k_1,k_2)}|^q} \sim AN\left(E|d_{0,(k_1,k_2)}|^q, \frac{V|d_{0,(k_1,k_2)}|^q}{2^{2j}} \right)$$

and from (17),

$$\overline{|d_{j,(k_1,k_2)}|^q} \sim AN\left(2^{-jHq} E|d_{0,(k_1,k_2)}|^q, 2^{-2jHq} \frac{V|d_{0,(k_1,k_2)}|^q}{2^{2j}} \right)$$

Finally, by applying *Delta's method*,

$$\log_2 \overline{|d_{j,(k_1,k_2)}|^q} \sim AN\left(-jHq + \log_2 E|d_{0,(k_1,k_2)}|^q, 2^{-2jHq} \frac{V|d_{0,(k_1,k_2)}|^q}{2^{2j}} \left(\frac{1}{\ln 2 \ 2^{-jHq} E|d_{0,(k_1,k_2)}|^q} \right)^2 \right),$$

which simplifies to

$$\begin{aligned} \log_2 \overline{|d_{j,(k_1,k_2)}|^q} &\sim AN\left(-jHq + \log_2 E|d_{0,(k_1,k_2)}|^q, \frac{V|d_{0,(k_1,k_2)}|^q}{\ln^2 2 \ 2^{2j} E^2|d_{0,(k_1,k_2)}|^q} \right), \\ &\sim AN\left(-jHq + \log_2 E|d_{0,(k_1,k_2)}|^q, \frac{1}{\ln^2 2 \ 2^{2j}} \left(\pi \frac{\Gamma\left(\frac{1+2q}{2}\right)}{\Gamma^2\left(\frac{1+q}{2}\right)} - 1 \right) \right). \end{aligned}$$

If a weighted linear regression

$$\widehat{\tau}(q) := \sum_{j=j_1}^{j_2} \omega_j \log_2 \overline{|d_{j,(k_1,k_2)}|^q},$$

is proposed to estimate the partition function with $\sum \omega_j = 0$ and $\sum -j\omega_j = 1$ (see Gonçalves et al. 1998), then from (16), the estimate is asymptotically unbiased.

Finally,

$$V\left(\frac{\log_2 \overline{|d_{j_1,(k_1,k_2)}|^q} - \log_2 \overline{|d_{j_2,(k_1,k_2)}|^q}}{j_1 - j_2}\right) =$$

$$\frac{1}{(j_1-j_2)^2 \ln^2 2}\left(\pi \frac{\Gamma\left(\frac{1+2q}{2}\right)}{\Gamma^2\left(\frac{1+q}{2}\right)} - 1\right)\frac{1}{HA\left(2^{2j_1}, 2^{2j_2}\right)},$$

where HA is the harmonic mean. Since weights ω_{j_1,j_2} are inverse-proportional to the variance then

$$\omega_{j_1 j_2} \propto (j_1-j_2)^2 HA\left(2^{2j_1}, 2^{2j_2}\right).$$

It is somewhat surprising that the weights $\omega_{j_1 j_2}$ do not depend on the power q.

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