Estimation of the Hurst Exponent Using Trimean Estimators on Nondecimated Wavelet Coefficients

Chen Feng and Branislav Vidakovic

Appendix A Technical Proofs.

Proof of Theorem III.1.

Proof. A single wavelet coefficient in a non-decimated wavelet transform of a fBm of size N with Hurst exponent H is normally distributed, with variance depending on its level j, therefore, each pair $d_{j,k}$ and $d_{j,k+N/2}$ in mid-energy $D_{j,k}$ are assumed to be independent and follow the same normal distribution.

$$d_{j,k},\ d_{j,k+N/2} \sim \mathcal{N}\left(0, 2^{-(2H+1)j}\sigma^2\right).$$

Then the mid-energy is defined as

$$D_{j,k} = \frac{\left(d_{j,k}^2 + d_{j,k+N/2}^2\right)}{2},\ j = 1, .., J, \text{and } k = 1, ..., N/2,$$

and it can be readily shown that $D_{j,k}$ has exponential distribution with scale parameter $\lambda_j = \sigma^2 \cdot 2^{-(2H+1)j}$, i.e.,

$$f\left(D_{j,k}\right) = \lambda_j^{-1} e^{-\lambda_j^{-1} D_{j,k}},\ \text{for any } k = 1, .., N/2.$$

Therefore the ith subgroup $\left\{D_{j,i}, D_{j,i+M}, D_{j,i+2M}, ..., D_{j,(N/2-M+i)}\right\}$ are i.i.d. $\exp\left(\lambda_j^{-1}\right)$, and when applying general trimean estimator $\hat{\mu}_{j,i}$ on $\left\{D_{j,i}, D_{j,i+M}, D_{j,i+2M}, ..., D_{j,(N/2-M+i)}\right\}$, following the derivation in Section II, we have

$$\boldsymbol{\xi} = \left[\log\left(\frac{1}{1-p}\right)\lambda_j \quad \log(2)\lambda_j \quad \log\left(\frac{1}{p}\right)\lambda_j\right]^T,$$

and

$$\Sigma = \begin{bmatrix} \frac{p}{(1-p)}\lambda_j^2 & \frac{p}{(1-p)}\lambda_j^2 & \frac{p}{(1-p)}\lambda_j^2 \\ \frac{p}{(1-p)}\lambda_j^2 & \lambda_j^2 & \lambda_j^2 \\ \frac{p}{(1-p)}\lambda_j^2 & \lambda_j^2 & \frac{1-p}{p}\lambda_j^2 \end{bmatrix}_{3\times 3},$$

therefore, the asymptotic distribution of $\hat{\mu}_{j,i}$ is normal with mean

$$\begin{aligned}\mathbb{E}\left(\hat{\mu}_{j,i}\right) &= A \cdot \boldsymbol{x} \\ &= \left(\frac{\alpha}{2}\log\left(\frac{1}{p(1-p)}\right) + (1-\alpha)\log 2\right)\lambda_j \\ &\triangleq c(\alpha, p)\lambda_j,\end{aligned}$$

and variance

$$\begin{aligned}\text{Var}\left(\hat{\mu}_{j,i}\right) &= \frac{2M}{N} A\Sigma A^T \\ &= \frac{2M}{N}\left(\frac{\alpha(1-2p)(\alpha-4p)}{4p(1-p)} + 1\right)\lambda_j^2 \\ &\triangleq \frac{2M}{N} f(\alpha, p)\lambda_j^2.\end{aligned}$$

□

Proof of Theorem III.2.

Proof. We have stated that $D_{j,k} \sim \mathcal{E}xp\left(\lambda_j^{-1}\right)$ with scale parameter $\lambda_j = \sigma^2 \cdot 2^{-(2H+1)j}$, so that

$$f\left(D_{j,k}\right) = \lambda_j^{-1} e^{-\lambda_j^{-1} D_{j,k}},\ \text{for any } k = 1, .., N/2.$$

Let $y_{j,k} = \log\left(D_{j,k}\right)$ for any $j = 1, ..., J$ and $k = 1, ..., N/2$. The pdf and cdf of $y_{j,k}$ are

$$f\left(y_{j,k}\right) = \lambda_j^{-1} e^{-\lambda_j^{-1} e^{y_{j,k}}} e^{y_{j,k}},$$

and

$$F\left(y_{j,k}\right) = 1 - e^{-\lambda_j^{-1} e^{y_{j,k}}}.$$

The p-quantile can be obtained by solving $F\left(y_p\right) = 1 - e^{-\lambda_j^{-1} e^{y_p}} = p$, and $y_p = \log\left(-\lambda_j \log(1-p)\right)$. Then it can be shown that $f\left(y_p\right) = -(1-p)\log(1-p)$. When applying the general trimean estimator $\hat{\mu}_{j,i}$ on $\left\{\log\left(D_{j,i}\right), \log\left(D_{j,i+M}\right), ..., \log\left(D_{j,(N/2-M+i)}\right)\right\}$, following the derivation in Section II, we get

$$\boldsymbol{\xi} = \begin{bmatrix} \log\left(\log\left(\frac{1}{1-p}\right)\right) + \log\left(\lambda_j\right) \\ \log(\log 2) + \log\left(\lambda_j\right) \\ \log\left(\log\left(\frac{1}{p}\right)\right) + \log\left(\lambda_j\right) \end{bmatrix},$$

and

$$\Sigma = \begin{bmatrix} \frac{p}{(1-p)(\log(1-p))^2} & \frac{p}{(1-p)\log(1-p)\log\left(\frac{1}{2}\right)} & \frac{p}{(1-p)\log(1-p)\log p} \\ \frac{p}{(1-p)\log(1-p)\log\left(\frac{1}{2}\right)} & \frac{1}{(\log 2)^2} & \frac{1}{\log\left(\frac{1}{2}\right)\log p} \\ \frac{p}{(1-p)\log(1-p)\log p} & \frac{1}{\log\left(\frac{1}{2}\right)\log p} & \frac{1-p}{p(\log p)^2} \end{bmatrix},$$

thus, the asymptotic distribution of $\hat{\mu}_{j,i}$ is normal with mean

$$\begin{aligned}\mathbb{E}\left(\hat{\mu}_{j,i}\right) &= A \cdot \boldsymbol{\xi} \\ &= \frac{\alpha}{2}\log\left(\log\frac{1}{1-p} \cdot \log\frac{1}{p}\right) + (1-\alpha)\log(\log 2) + \log\left(\lambda_j\right) \\ &\triangleq c(\alpha, p) + \log\left(\lambda_j\right),\end{aligned}$$

and variance

$$\begin{aligned}\text{Var}\left(\hat{\mu}_{j,i}\right) &= \frac{1}{N/16} A\Sigma A^T \\ &= \frac{2M}{N}\left(\frac{\alpha^2}{4}g_1(p) + \frac{\alpha(1-\alpha)}{2}g_2(p) + \frac{(1-\alpha)^2}{(\log 2)^2}\right) \\ &\triangleq \frac{2M}{N} f(\alpha, p),\end{aligned}$$

where

$$g_1(p) = \frac{p}{(1-p)(\log(1-p))^2} + \frac{1-p}{p(\log p)^2} + \frac{2p}{(1-p)\log(1-p)\log p},$$

and

$$g_2(p) = \frac{2p}{(1-p)\log(1-p)\log\frac{1}{2}} + \frac{2}{\log\frac{1}{2}\log p}.$$

□

Proof of Lemma 2.

Proof. When applying Tukey's trimean estimator $\hat{\mu}_{j,i}^{T}$ on $\left\{D_{j,i}, D_{j,i+M}, D_{j,i+2M}, ..., D_{j,(N/2-M+i)}\right\}$, following the derivation in Section II-A, we have

$$\boldsymbol{\xi}_T = \begin{bmatrix} \log\left(\frac{4}{3}\right)\lambda_j \\ \log\left(2\right)\lambda_j \\ \log\left(4\right)\lambda_j \end{bmatrix},$$

and

$$\Sigma_T = \begin{bmatrix} \frac{1}{3}\lambda_j^2 & \frac{1}{3}\lambda_j^2 & \frac{1}{3}\lambda_j^2 \\ \frac{1}{3}\lambda_j^2 & \lambda_j^2 & \lambda_j^2 \\ \frac{1}{3}\lambda_j^2 & \lambda_j^2 & \frac{1}{3}\lambda_j^2 \end{bmatrix}_{3\times 3},$$

therefore, the asymptotic distribution of $\hat{\mu}_{j,i}^{T}$ is normal with mean

$$\mathbb{E}\left(\hat{\mu}_{j,i}^{T}\right) = A_T \cdot \boldsymbol{\xi}_T = \frac{1}{4}\log\left(\frac{64}{3}\right)\lambda_j \triangleq c_1\lambda_j,$$

and variance

$$\mathrm{Var}\left(\hat{\mu}_{j,i}^{T}\right) = \frac{2M}{N}A_T\Sigma_T A_T^T = \frac{5M}{3N}\lambda_j^2.$$

When applying Gastwirth estimator $\hat{\mu}_{j,i}^{G}$ on $\left\{D_{j,i}, D_{j,i+M}, D_{j,i+2M}, ..., D_{j,(N/2-M+i)}\right\}$, following the derivation in Section II-B, we have

$$\boldsymbol{\xi}_G = \begin{bmatrix} \log\left(\frac{3}{2}\right)\lambda_j \\ \log\left(2\right)\lambda_j \\ \log\left(3\right)\lambda_j \end{bmatrix},$$

and

$$\Sigma_G = \begin{bmatrix} \frac{1}{2}\lambda_j^2 & \frac{1}{2}\lambda_j^2 & \frac{1}{2}\lambda_j^2 \\ \frac{1}{2}\lambda_j^2 & \lambda_j^2 & \lambda_j^2 \\ \frac{1}{2}\lambda_j^2 & \lambda_j^2 & \frac{1}{2}\lambda_j^2 \end{bmatrix},$$

therefore, the asymptotic distribution of $\hat{\mu}_{j,i}^{G}$ is normal with mean

$$\begin{aligned}\mathbb{E}\left(\hat{\mu}_{j,i}^{G}\right) &= A_G \cdot \boldsymbol{\xi}_G \\ &= \left(0.3\times\log\left(\frac{9}{2}\right) + 0.4\times\log\left(2\right)\right)\lambda_j \\ &\triangleq c_2\lambda_j,\end{aligned}$$

and variance

$$\mathrm{Var}\left(\hat{\mu}_{j,i}^{G}\right) = \frac{2M}{N}A_G\Sigma_G A_G^T = \frac{1.67M}{N}\lambda_j^2.$$

□

Proof of Lemma 3.

Proof. When applying Tukey's trimean estimator $\hat{\mu}_{j,i}^{T}$ on $\left\{\log\left(D_{j,i}\right), \log\left(D_{j,i+M}\right), ..., \log\left(D_{j,(N/2-M+i)}\right)\right\}$, following the derivation in Section II-A, we have

$$\boldsymbol{\xi}_T = \begin{bmatrix} \log\left(\log\left(\frac{4}{3}\right)\right) + \log\left(\lambda_j\right) \\ \log\left(\log 2\right) + \log\left(\lambda_j\right) \\ \log\left(\log 4\right) + \log\left(\lambda_j\right) \end{bmatrix},$$

and

$$\Sigma_T = \begin{bmatrix} \frac{1}{3\left(\log\left(\frac{3}{4}\right)\right)^2} & \frac{1}{3\log\left(\frac{3}{4}\right)\log\left(\frac{1}{2}\right)} & \frac{1}{3\log\left(\frac{3}{4}\right)\log\left(\frac{1}{4}\right)} \\ \frac{1}{3\log\left(\frac{3}{4}\right)\log\left(\frac{1}{2}\right)} & \frac{1}{(\log 2)^2} & \frac{1}{\log\left(\frac{1}{2}\right)\log\left(\frac{1}{4}\right)} \\ \frac{1}{3\log\left(\frac{3}{4}\right)\log\left(\frac{1}{4}\right)} & \frac{1}{\log\left(\frac{1}{2}\right)\log\left(\frac{1}{4}\right)} & \frac{3}{(\log 4)^2} \end{bmatrix},$$

therefore, the asymptotic distribution of $\hat{\mu}_{j,i}^{T}$ is normal with mean

$$\begin{aligned}\mathbb{E}\left(\hat{\mu}_{j,i}^{T}\right) &= A_T \cdot \boldsymbol{\xi}_T \\ &= -\left(2H+1\right)\log 2\cdot j + \log\sigma^2 + \\ &\quad \frac{1}{4}\log\left(\log\left(\frac{4}{3}\right)\cdot\log 4\right) + \frac{1}{2}\log\left(\log 2\right) \\ &\triangleq -\left(2H+1\right)\log 2\cdot j + c_3\end{aligned}$$

and variance

$$\begin{aligned}\mathrm{Var}\left(\hat{\mu}_{j,i}^{T}\right) &= \frac{2M}{N}A_T\Sigma_T A_T^T \\ &= \frac{M}{2N}\left(\frac{1}{12\left(\log\frac{3}{4}\right)^2} + \frac{1}{3\log\frac{3}{4}\log\frac{1}{2}} + \frac{1}{6\log\frac{3}{4}\log\frac{1}{4}} + \right. \\ &\quad \left. \frac{1}{\left(\log\frac{1}{2}\right)^2} + \frac{1}{\log\frac{1}{2}\log\frac{1}{4}} + \frac{3}{4\left(\log\frac{1}{4}\right)^2}\right) \\ &\triangleq V_1.\end{aligned}$$

When applying Gastwirth estimator $\hat{\mu}_{j,i}^{G}$ on $\left\{\log\left(D_{j,i}\right), \log\left(D_{j,i+M}\right), ..., \log\left(D_{j,(N/2-M+i)}\right)\right\}$, following the derivation in Section II-B, we have

$$\boldsymbol{\xi}_G = \begin{bmatrix} \log\left(\log\left(\frac{3}{2}\right)\right) + \log\left(\lambda_j\right) \\ \log\left(\log 2\right) + \log\left(\lambda_j\right) \\ \log\left(\log 3\right) + \log\left(\lambda_j\right) \end{bmatrix},$$

and

$$\Sigma_G = \begin{bmatrix} \frac{1}{2\left(\log\frac{2}{3}\right)^2} & \frac{1}{2\log\left(\frac{2}{3}\right)\log\left(\frac{1}{2}\right)} & \frac{1}{2\log\left(\frac{1}{3}\right)\log\left(\frac{2}{3}\right)} \\ \frac{1}{2\log\left(\frac{2}{3}\right)\log\left(\frac{1}{2}\right)} & \frac{1}{(\log 2)^2} & \frac{1}{\log\left(\frac{1}{2}\right)\log\left(\frac{1}{3}\right)} \\ \frac{1}{2\log\left(\frac{1}{3}\right)\log\left(\frac{2}{3}\right)} & \frac{1}{\log\left(\frac{1}{2}\right)\log\left(\frac{1}{3}\right)} & \frac{2}{(\log 3)^2} \end{bmatrix},$$

therefore, the asymptotic distribution of $\hat{\mu}_{j,i}^{G}$ is normal with mean

$$\begin{aligned}\mathbb{E}\left(\hat{\mu}_{j,i}^{G}\right) &= A_g \cdot \boldsymbol{\xi}_G \\ &= -\left(2H+1\right)\log 2\cdot j + \log\sigma^2 + \\ &\quad 0.3\times\log\left(\log\left(\frac{3}{2}\right)\cdot\log 3\right) + 0.4\times\log\left(\log 2\right) \\ &\triangleq -\left(2H+1\right)\log 2\cdot j + c_4\end{aligned}$$

and variance

$$\begin{aligned}\mathrm{Var}\left(\hat{\mu}_{j,i}^{G}\right) &= \frac{2M}{N}A_G\Sigma_G A_G^T \\ &= \frac{2M}{N}\left(\frac{0.09}{2\left(\log\frac{2}{3}\right)^2} + \frac{0.12}{\log\frac{2}{3}\log\frac{1}{2}} + \frac{0.09}{\log\frac{1}{3}\log\frac{2}{3}} + \right. \\ &\quad \left. \frac{0.16}{\left(\log\frac{1}{2}\right)^2} + \frac{0.24}{\log\frac{1}{2}\log\frac{1}{3}} + \frac{0.18}{\left(\log\frac{1}{3}\right)^2}\right) \\ &\triangleq V_2.\end{aligned}$$

□

Appendix B
Additional Simulation Results

TABLE I: Simulation Results for $N = 2^{10}$ fBm(H) using Haar wavelet (300 Replications)

	Methods										
H	VA	SSB	MEDL	MEDLA	TT	TTME	TTLME	GME	GLME	GTME	GTLME
					$\hat{H}$						
0.3000	0.2508	0.2496	0.2445	0.2428	0.2162	0.2461	0.2465	0.2455	0.2459	0.2456	0.2466
0.4000	0.3720	0.3716	0.3618	0.3625	0.3515	0.3670	0.3661	0.3656	0.3652	0.3661	0.3666
0.5000	0.4880	0.4916	0.4777	0.4802	0.4885	0.4860	0.4836	0.4833	0.4823	0.4846	0.4841
0.6000	0.5981	0.6098	0.5910	0.5905	0.6336	0.5981	0.5965	0.5955	0.5950	0.5968	0.5970
0.7000	0.7031	0.7257	0.7016	0.7031	0.7817	0.7100	0.7083	0.7079	0.7070	0.7089	0.7088
					Variances						
0.3000	0.0030	0.0023	0.0024	0.0021	**0.0020**	**0.0020**	0.0021	0.0021	0.0021	0.0021	0.0021
0.4000	0.0032	0.0026	0.0027	0.0024	0.0030	**0.0022**	0.0024	0.0024	0.0024	0.0023	0.0024
0.5000	0.0033	0.0028	0.0032	0.0024	0.0047	**0.0023**	0.0025	0.0025	0.0025	0.0024	0.0025
0.6000	0.0049	0.0041	0.0038	**0.0031**	0.0097	**0.0031**	0.0033	0.0032	0.0033	**0.0031**	0.0033
0.7000	0.0068	0.0050	0.0046	0.0033	0.0182	**0.0032**	0.0035	0.0034	0.0036	0.0033	0.0035
					MSEs						
0.3000	0.0054	**0.0048**	0.0055	0.0054	0.0090	0.0049	0.0050	0.0050	0.0051	0.0050	0.0049
0.4000	0.0040	0.0034	0.0042	0.0038	0.0054	**0.0033**	0.0035	0.0035	0.0037	0.0034	0.0035
0.5000	0.0035	0.0029	0.0037	0.0028	0.0049	**0.0025**	0.0027	0.0027	0.0028	0.0026	0.0027
0.6000	0.0049	0.0042	0.0038	0.0032	0.0108	**0.0031**	0.0033	0.0032	0.0033	**0.0031**	0.0033
0.7000	0.0068	0.0057	0.0046	**0.0033**	0.0248	**0.0033**	0.0036	0.0035	0.0036	0.0034	0.0036

TABLE II: Simulation Results for $N = 2^{10}$ fBm(H) using Daubechies 6 wavelet (300 Replications)

	Methods										
H	VA	SSB	MEDL	MEDLA	TT	TTME	TTLME	GME	GLME	GTME	GTLME
					$\hat{H}$						
0.3000	0.2468	0.2505	0.2461	0.2453	0.2084	0.2508	0.2501	0.2495	0.2491	0.2502	0.2505
0.4000	0.3709	0.3820	0.3734	0.3724	0.3443	0.3807	0.3793	0.3788	0.3782	0.3794	0.3799
0.5000	0.4765	0.5052	0.4894	0.4909	0.4649	0.5054	0.5017	0.5019	0.5002	0.5030	0.5024
0.6000	0.5673	0.6317	0.6055	0.6084	0.5844	0.6329	0.6241	0.6250	0.6216	0.6278	0.6258
0.7000	0.6454	0.7667	0.7236	0.7278	0.6999	0.7707	0.7543	0.7547	0.7486	0.7599	0.7576
					Variances						
0.3000	0.0027	0.0026	0.0026	0.0022	**0.0014**	0.0025	0.0025	0.0025	0.0025	0.0025	0.0025
0.4000	0.0022	0.0024	0.0025	0.0021	**0.0016**	0.0024	0.0024	0.0023	0.0024	0.0023	0.0024
0.5000	**0.0023**	0.0033	0.0028	0.0025	0.0027	0.0031	0.0029	0.0028	0.0028	0.0029	0.0029
0.6000	**0.0020**	0.0043	0.0030	0.0028	0.0038	0.0043	0.0035	0.0036	0.0033	0.0038	0.0037
0.7000	**0.0018**	0.0059	0.0029	0.0026	0.0058	0.0064	0.0044	0.0044	0.0039	0.0049	0.0047
					MSEs						
0.3000	0.0055	0.0051	0.0055	0.0052	0.0098	**0.0049**	0.0050	0.0050	0.0051	**0.0049**	0.0050
0.4000	0.0031	0.0028	0.0032	0.0029	0.0047	**0.0027**	0.0028	0.0028	0.0028	0.0028	0.0028
0.5000	0.0028	0.0033	0.0029	**0.0025**	0.0040	0.0031	0.0029	0.0028	0.0028	0.0029	0.0029
0.6000	0.0031	0.0053	0.0030	**0.0028**	0.0040	0.0054	0.0041	0.0042	0.0038	0.0046	0.0043
0.7000	0.0047	0.0104	**0.0034**	**0.0034**	0.0058	0.0114	0.0073	0.0074	0.0062	0.0085	0.0080

TABLE III: Simulation Results for $N = 2^{10}$ fBm(H) using Symmlet 8 wavelet (300 Replications)

	Methods										
H	VA	SSB	MEDL	MEDLA	TT	TTME	TTLME	GME	GLME	GTME	GTLME
					$\hat{H}$						
0.3000	0.2441	0.2513	0.2483	0.2520	0.2070	0.2506	0.2511	0.2504	0.2507	0.2504	0.2513
0.4000	0.3684	0.3840	0.3757	0.3804	0.3425	0.3817	0.3813	0.3809	0.3804	0.3813	0.3819
0.5000	0.4720	0.5085	0.4972	0.5051	0.4602	0.5082	0.5064	0.5060	0.5052	0.5071	0.5070
0.6000	0.5541	0.6393	0.6160	0.6314	0.5679	0.6394	0.6339	0.6337	0.6315	0.6363	0.6353
0.7000	0.6134	0.7771	0.7375	0.7640	0.6545	0.7827	0.7705	0.7710	0.7662	0.7762	0.7731
					Variances						
0.3000	0.0028	0.0030	0.0029	0.0028	**0.0014**	0.0027	0.0028	0.0027	0.0028	0.0027	0.0028
0.4000	0.0023	0.0028	0.0030	0.0026	**0.0015**	0.0026	0.0027	0.0026	0.0027	0.0026	0.0027
0.5000	**0.0021**	0.0038	0.0033	0.0032	0.0023	0.0035	0.0033	0.0033	0.0032	0.0034	0.0034
0.6000	**0.0016**	0.0051	0.0036	0.0041	0.0022	0.0049	0.0043	0.0043	0.0041	0.0045	0.0044
0.7000	**0.0013**	0.0075	0.0036	0.0055	0.0017	0.0083	0.0062	0.0063	0.0056	0.0072	0.0066
					MSEs						
0.3000	0.0059	0.0053	0.0056	**0.0051**	0.0100	0.0052	0.0052	0.0052	0.0052	**0.0051**	0.0052
0.4000	0.0033	0.0031	0.0035	0.0030	0.0048	0.0030	0.0030	0.0030	0.0030	**0.0029**	0.0030
0.5000	**0.0029**	0.0039	0.0033	0.0032	0.0038	0.0035	0.0033	0.0033	0.0033	0.0034	0.0034
0.6000	0.0037	0.0066	0.0038	0.0051	**0.0032**	0.0064	0.0054	0.0054	0.0051	0.0058	0.0057
0.7000	0.0088	0.0134	0.0050	0.0096	**0.0038**	0.0151	0.0112	0.0113	0.0100	0.0129	0.0119