If the length of the data set is not a power of 2, but of the form $M \cdot 2^K$, for M odd and K a positive integer, then only K steps in the decomposition algorithm can be performed. For precise descriptions of conceptual and calculational hurdles caused by boundaries and data sets whose lengths are not a power of 2, we direct the reader to Bruce and Gao [42], Taswell and McGill [399], and monograph by Wickerhauser [457].

4.3.1 Discrete Wavelet Transformations as Linear Transformations

The change of basis in V_1 from $\mathcal{B}_1 = \{\phi_{1k}(x), k \in Z\}$ to $\mathcal{B}_2 = \{\phi_{0k}, k \in Z\} \cup \{\psi_{0k}, k \in Z\}$ can be performed by matrix multiplication. Therefore, it is possible to define discrete wavelet transformation by matrices. We have already seen a transformation matrix corresponding to Haar's inverse transformation in Example 4.1.2.

Let the length of the input signal be 2^J, let $\underset{\sim}{h} = \{h_s,\ s \in \mathbb{Z}\}$ be the wavelet filter, and let N be an appropriately chosen constant.

Denote by H_k a matrix of size $(2^{J-k} \times 2^{J-k+1}),\ k = 1, \ldots$ with entries

$$h_s, \quad s = (N-1) + (i-1) - 2(j-1) \textit{ modulo } 2^{J-k+1}, \tag{4.8}$$

at the position (i, j).

Note that H_k is a circulant matrix, its ith row is 1st row circularly shifted to the right by $2(i-1)$ units. This circularity is a consequence of using the *modulo* operator in (4.8).

By analogy, define a matrix G_k by using the filter $\underset{\sim}{g}$. A version of G_k corresponding to the already defined H_k can be obtained by changing h_i by $(-1)^i h_{N+1-i}$. The constant N is a shift parameter and affects the position of the wavelet on the time scale. For filters from the Daubechies family, a standard choice for N is the number of vanishing moments. See also Remark 3.3.4.

The matrix $\begin{bmatrix} H_k \\ G_k \end{bmatrix}$ is a basis-changing matrix in 2^{J-k+1} dimensional space; consequently, it is unitary.

Therefore,

$$I_{2^{J-k}} = [H_k' \; G_k'] \begin{bmatrix} H_k \\ G_k \end{bmatrix} = H_k' \cdot H_k + G_k' \cdot G_k.$$

and

$$I = \begin{bmatrix} H_k \\ G_k \end{bmatrix} \cdot [H_k' \; G_k'] = \begin{bmatrix} H_k \cdot H_k' & H_k \cdot G_k' \\ G_k \cdot H_k' & G_k \cdot G_k' \end{bmatrix}.$$

This implies,

$H_k \cdot H_k' = I,\ G_k \cdot G_k' = I,\ G_k \cdot H_k' = H_k \cdot G_k' = 0,$ and $H_k' \cdot H_k + G_k' \cdot G_k = I.$

Now, for a sequence y the J-step wavelet transformation is $\underset{\sim}{d} = W_J \cdot \underset{\sim}{y}$, where

$$W_1 = \begin{bmatrix} H_1 \\ G_1 \end{bmatrix},\ W_2 = \begin{bmatrix} \begin{bmatrix} H_2 \\ G_2 \end{bmatrix} \cdot H_1 \\ G_1 \end{bmatrix},$$
$$W_3 = \begin{bmatrix} \begin{bmatrix} \begin{bmatrix} H_3 \\ G_3 \end{bmatrix} \cdot H_2 \\ G_2 \end{bmatrix} \cdot H_1 \\ G_1 \end{bmatrix}, \ldots$$

Example 4.3.2 Suppose that $\underset{\sim}{y} = \{1, 0, -3, 2, 1, 0, 1, 2\}$ and the filter is $\underset{\sim}{h} = (h_0, h_1, h_2, h_3) = \left(\frac{1+\sqrt{3}}{4\sqrt{2}}, \frac{3+\sqrt{3}}{4\sqrt{2}}, \frac{3-\sqrt{3}}{4\sqrt{2}}, \frac{1-\sqrt{3}}{4\sqrt{2}}\right)$. Then, $J = 3$ and the matrices H_k and G_k are of dimension $2^{3-k} \times 2^{3-k+1}$.

$$H_1 = \begin{bmatrix} h_1 & h_2 & h_3 & 0 & 0 & 0 & 0 & h_0 \\ 0 & h_0 & h_1 & h_2 & h_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & h_0 & h_1 & h_2 & h_3 & 0 \\ h_3 & 0 & 0 & 0 & 0 & h_0 & h_1 & h_2 \end{bmatrix}$$
$$G_1 = \begin{bmatrix} -h_2 & h_1 & -h_0 & 0 & 0 & 0 & 0 & h_3 \\ 0 & h_3 & -h_2 & h_1 & -h_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & h_3 & -h_2 & h_1 & -h_0 & 0 \\ -h_0 & 0 & 0 & 0 & 0 & h_3 & -h_2 & h_1 \end{bmatrix}.$$

Since,

$$\begin{aligned} H_1 \cdot \underset{\sim}{y} &= \{2.19067, -2.19067, 1.67303, 1.15539\} \\ G_1 \cdot \underset{\sim}{y} &= \{0.96593, 1.86250, -0.96593, 0.96593\}. \end{aligned}$$

$W_1 \underset{\sim}{y} = \{2.19067, -2.19067, 1.67303, 1.15539 \mid 0.96593, 1.86250, -0.96593, 0.96593\}.$

$$H_2 = \begin{bmatrix} h_1 & h_2 & h_3 & h_0 \\ h_3 & h_0 & h_1 & h_2 \end{bmatrix} \quad G_2 = \begin{bmatrix} -h_2 & h_1 & -h_0 & h_3 \\ -h_0 & h_3 & -h_2 & h_1 \end{bmatrix}.$$

In this example, due to the lengths of the filter and the data, we can perform the transformation for two steps only, W_1 and W_2.

The two-step DAUB2 discrete wavelet transformation of $\underset{\sim}{y}$ is
$W_2 \cdot \underset{\sim}{y} = \{1.68301, 0.31699 \mid -3.28109, -0.18301 \mid 0.96593, 1.86250, -0.96593, 0.96593\},$

because

$$\begin{aligned}
H_2 \cdot H_1 \cdot \underset{\sim}{y} &= H_2 \cdot \{2.19067, -2.19067, 1.67303, 1.15539\} \\
&= \{1.68301, 0.31699\} \\
G_2 \cdot H_1 \cdot \underset{\sim}{y} &= G_1 \cdot \{2.19067, -2.19067, 1.67303, 1.15539\} \\
&= \{-3.28109, -0.18301\}.
\end{aligned}$$

The S+WAVELETS command
$>$ `dwt(c(1, 0, -3, 2, 1, 0, 1, 2), wavelet="d4", n.levels=2)`
produces the same output.

4.4 EXERCISES

4.1. Let $\underset{\sim}{h} = \{1, 2, 1\}$ and $\underset{\sim}{y} = \{1, 0, -3, 2, 1, 0, 1, 2\}$. Find $H\underset{\sim}{y}$ and $H^2\underset{\sim}{y}$.

4.2. What are the columns in the 8 $\times 8$ matrix in Example 4.1.2?

4.3. Using the cascade algorithm, find $\psi(\frac{1}{2}), \psi(1)$, and $\phi(\frac{1}{4})$ in Example 4.2.1.

4.4. Show that for the Haar transformation and data $\{y_1, \ldots, y_{2^n}\},\ c_{00} = (\sqrt{2})^n \bar{y}$.

4.5. By mimicking the procedures in Fig. 4.6 and 4.7, perform Haar's direct transformation on the sequence $\{1, 2, \ldots, 16\}$. Perform the inverse Haar transformation on {34, -16, $-4\sqrt{2}$, $-4\sqrt{2}$,-2,-2,-2,-2, $-\sqrt{2}/2$, $-\sqrt{2}/2$,$-\sqrt{2}/2$, $-\sqrt{2}/2$,$-\sqrt{2}/2$, $-\sqrt{2}/2$,$-\sqrt{2}/2$,$-\sqrt{2}/2\}$.