An Open Problem or Easy Exercise

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Let $\mathbf{h} = \{h_0, h_1, \dots, h_{2n-1}\}$ be any orthogonal wavelet filter of length $2n$. Define k-th autocorrelation coefficient of filter $\mathbf{h}$ as

$$a_k^{(2n)} = \sum_{i\in\mathbb{Z}} h_{i+k}h_i, \ k \in \mathbb{Z}.$$

Of course, for any n, $a_0^{(2n)} = 1$, $a_{2m}^{(2n)} = 0, m = 1, 2, \dots$ (orthogonality), and $\sum_k a_k^{(2n)} = 2$ (since $\sum_k h_k = \sqrt{2}$ and $a_{-k}^{(2n)} = a_k^{(2n)}$).

(i) Show that for Daubechies' extremal phase and Daubechies' least asymmetric families (Daublets and Symmlets) for any n the values $a_k^{(2n)}$ coincide at any $k \in \mathbb{Z}$. Nothing to prove for $n = 1, 2,$ and 3, since Daubechies' and Symmlet filter taps coincide.

(Solution) Nothing to prove for $n > 3$ either. For any n, Daublets and Symmlets share the trigonometric polynomial $|m_0(\omega)|^2$ by construction. The difference between them is in the selection of a square root of $|m_0(\omega)|^2$, the function $m_0(\omega) = \frac{1}{\sqrt{2}}\sum_k h_k e^{-ik\omega}$.

The function $|m_0(\omega)|^2$ is a trigonometric polynomial that can be expressed in terms of cosines as

$$|m_0(\omega)|^2 = 1/2 + \sum_{k=1}^{n} a_{2k-1}^{(2n)} \cos(2k-1)\omega,$$

where $a_{2k-1}^{(2n)}$ are the autocorrelation coefficients. This is a consequence of straightforward rearrangement of sums in $m_0(\omega)\overline{m_0(\omega)}$, use of elementary properties of trigonometric functions and orthogonality.

For instance, Daubechies 4 filter $(\frac{1+\sqrt{3}}{4\sqrt{2}}, \frac{3+\sqrt{3}}{4\sqrt{2}}, \frac{3-\sqrt{3}}{4\sqrt{2}}, \frac{1-\sqrt{3}}{4\sqrt{2}})$, has $|m_0(\omega)|^2 = (\cos^2\frac{\omega}{2})^2|\mathbb{L}(\omega)|^2$, where $|\mathbb{L}(\omega)|^2 = \sum_{k=0}^{2-1}\binom{2+k-1}{k}\sin^{2k}\frac{\omega}{2} = 2 - \cos\omega$. Using the identifies $\cos^2\frac{\omega}{2} = \frac{1+\cos\omega}{2}$, and $\cos^3\omega = \frac{1}{4}[\cos(3\omega) + 3\cos\omega]$ one gets $|m_0(\omega)|^2 = \frac{1}{2} + \frac{9}{16}\cos\omega - \frac{1}{16}\cos(3\omega)$. Indeed, $a_1 = 9/16$ and $a_3 = -1/16$.

(ii) **Prove/Disprove:** For a fixed family of wavelets (Daubechies', Symmlets, or Coiflets), points $(\frac{1}{2n}, a_1^{(2n)}),\ n = 3, 4, 5, \dots$ fall on a straight line, see Figure 1.

(iii) **Prove/Disprove:** For a fixed family of wavelets (Daubechies', Symmlets, or Coiflets), $\lim_{n\to\infty} a_{2m+1}^{(2n)} = a_{2m+1}^{(\infty)} = (-1)^m\frac{2}{(2m+1)\pi},\ m = 0, 1, 2, \dots.$

Remark 1. The fact $\sum_k a_k^{(2n)} = 2$ is in a limiting agreement with (iii) since $\arctan(1) = \frac{\pi}{4} = \sum_{m=0}^{\infty}(-1)^m\frac{1}{(2m+1)}$, $a_k^{(2n)} = a_{-k}^{(2n)}$, and $a_0^{(\infty)} + \sum_{m=0}^{\infty} a_{2m+1}^{(\infty)} + \sum_{m=0}^{\infty} a_{-(2m+1)}^{(\infty)} = 2.$

Remark 2. For the Shannon wavelet the identity $a_{2m+1}^{(\infty)} = (-1)^m\frac{2}{(2m+1)\pi},\ m = 0, 1, 2, \dots$ is

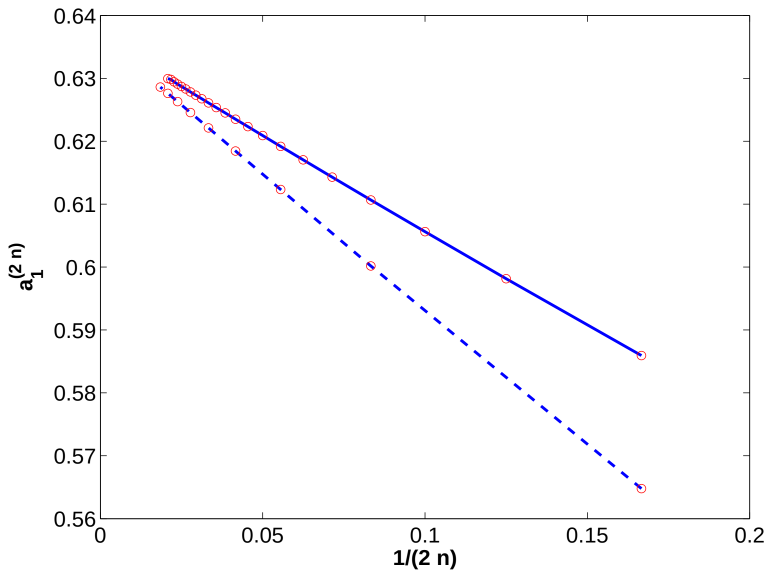

Figure 1: Values $a_1^{(2n)}$ plotted against the reciprocal of filter length, $1/(2n)$. Daubechies' (and Symmlet, see (i)) values for $a_1^{(2n)}$ fall on the solid line, for the Coiflet family the values fall on the dotted line.

simple. Indeed, since $h_k = \frac{1}{\sqrt{2}}\text{sinc}(k/2) = \frac{\sqrt{2}}{k\pi}\sin\frac{k\pi}{2}$, $a_{2m+1}^{\infty} = 2h_0h_{2m+1} = 2\cdot\frac{1}{\sqrt{2}}\cdot\frac{\sqrt{2}(-1)^m}{(2m+1)\pi} = \frac{2(-1)^m}{(2m+1)\pi}$.

Not all infinite orthogonal wavelet filters satisfy $a_{2m+1}^{(\infty)} = (-1)^m\frac{2}{(2m+1)\pi},\ m = 0,1,2,\dots$. For example, the filter for standard Meyer wavelet (taper function $\nu(x) = x$) is infinite and symmetric. Its taps are given by

$$h_k = \frac{\sqrt{2}}{k\pi(9-4k^2)}\left(9\sin\frac{k\pi}{3} + 6k\cos\frac{2k\pi}{3}\right), \quad k \in \mathbb{Z}.$$

In this case, $a_1^{\infty} = 0.620245007349516 \pm 1/2\cdot 10^{-15} < 2/\pi$. It is likely, however, that when the taper degree increases ($\nu_1(x) = x, \nu_2(x) = x^2(3-2x), \nu_3(x) = x^3(10-15x+6x^2), \dots, \nu_s(x) = \frac{B(x,s,s)}{B(s,s)}, \dots$), $a_1^{\infty}(s) \to 2/\pi,\ \ s \to \infty$.

Remark 3. Can a wavelet be constructed so that $a_1 = 0$? Yes. Here is an example.

Consider the family of filters, call it GT,

$$h_0 = -\frac{\sqrt{2}}{2b}, h_1 = \frac{\sqrt{2}}{2}, h_{2k} = \frac{(b^2-1)\sqrt{2}}{2b^{k+1}},\ k = 1,2,\dots, h_{2k+1} = 0,\ k = \pm 1, \pm 2, \dots.$$

• Any $|b| > 1$ gives a valid ON wavelet filter. Wavelets are not of compact support, but their decay is exponential. However, if $|b|$ is close to 1, the basis is not practical since the taps decay to 0 slowly.

• For $b = \frac{1+\sqrt{5}}{2}$ (any connection with Fibonacci sequences?) the a_1 equals 0. What does it mean in terms of wavelet/scaling functions? If $a_1 = 0$, an additional layer of orthogonality is

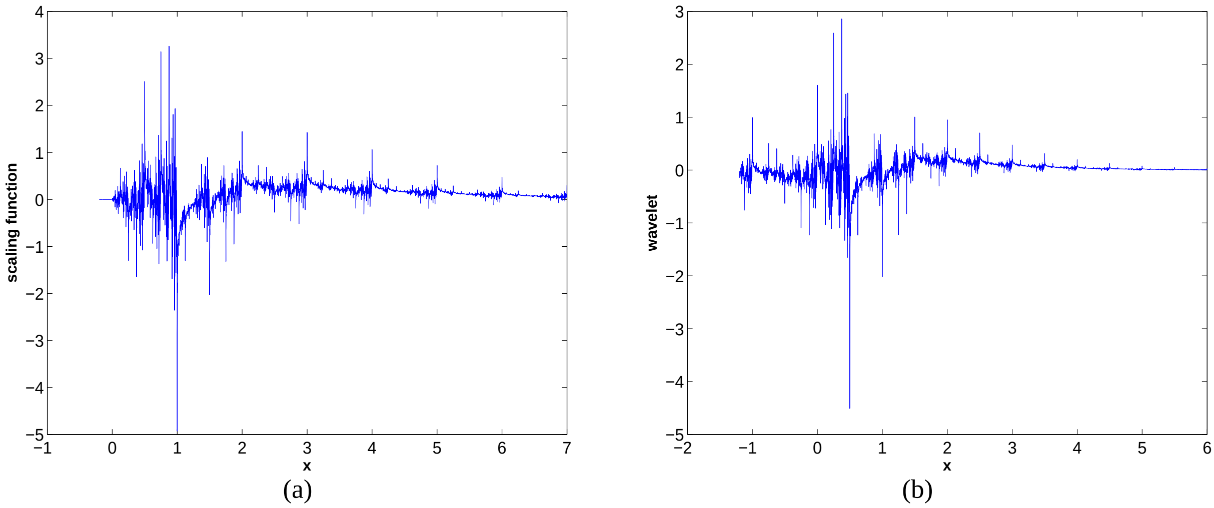

Figure 2: GT Scaling function and Wavelet for $b = \frac{1+\sqrt{5}}{2}$.

imposed: the scaling function $\phi(x)$ is orthogonal not only to its integer shifts, but to its immediate 1/2 shifts, i.e., $\phi(x - 1/2)$ and $\phi(x + 1/2)$.

• Is the wavelet, as given in Figure 2(a,b) *good for anything*? Being self-similar, the scaling function looks comparably "bad," at any resolution. It turns out that GT wavelet is good for estimating the Hurst exponent of monofractals with low regularity (H close to 0). Such signals are nasty and require a nasty wavelet. Something in the spirit, *a vaccine for a disease is made of disabled bacteria responsible for the disease.*

• If $b = -3$ the wavelet has 3 vanishing moments, i.e., $\sum_{k=0}^{\infty}(-1)^k k^m h_k = 0$, for $m = 0, 1, 2$. This means that the wavelet is continuously differentiable (smooth). Figure 3 shows the scaling function and the wavelet; they look as Daubechies' extremal phase wavelets with 4 or 5 vanishing moments. In this case, $a_1 = 11/18$, quite close to $2/\pi$, $a_3 = -4/27$, etc.

Remark 4. Artistic patterns made by $a_{2k-1}^{(2n)}$'s.

Pollen family is an interesting generator of wavelets that are indexed by two angles, $\varphi_1, \varphi_2 \in [0, 2\pi]$. The taps of Pollen filters are given in Table 1. In this case, $a_{2k-1}^{(2n)}$'s are trigonometric functions if explicit and simple form.

Figure 4 shows $a_1^{(6)}, a_3^{(6)}$ and $a_5^{(6)}$ for the two parameter Pollen family for $0 \leq \varphi_1 \leq 4\pi$ and $0 \leq \varphi_2 \leq 4\pi$.

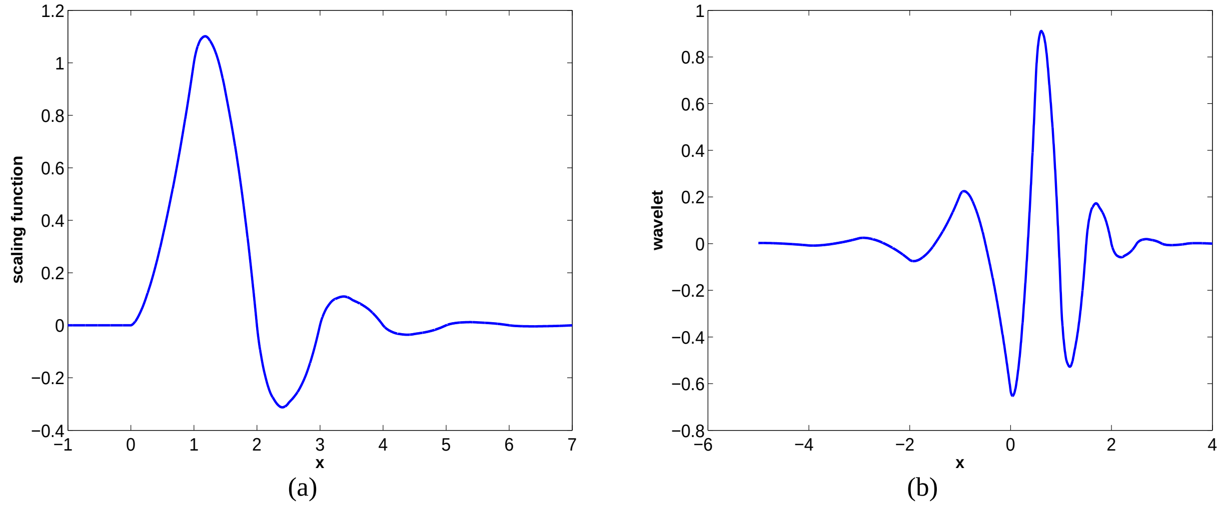

Figure 3: Smooth GT scaling function and wavelet, $b = -3$.

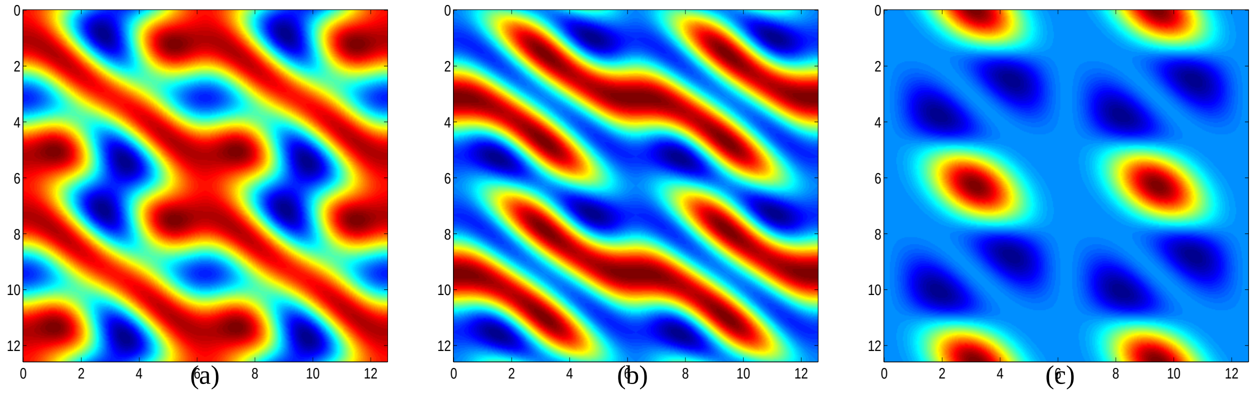

Figure 4: Trigonometric functions $a_1^{(6)}$, $a_3^{(6)}$ and $a_5^{(6)}$ for the two parameter Pollen family.

Table 1: Pollen parameterization for six-tap filters [$s = 2\sqrt{2}$].

n	h_n
0	$(1 + \cos\varphi_1 - \cos\varphi_2 - \cos\varphi_1\cos\varphi_2 + \sin\varphi_1 - \cos\varphi_2\sin\varphi_1 - \sin\varphi_2$ $+ \cos\varphi_1\sin\varphi_2 - \sin\varphi_1\sin\varphi_2)/(2s)$
1	$(1 - \cos\varphi_1 + \cos\varphi_2 - \cos\varphi_1\cos\varphi_2 + \sin\varphi_1 + \cos\varphi_2\sin\varphi_1 - \sin\varphi_2$ $- \cos\varphi_1\sin\varphi_2 - \sin\varphi_1\sin\varphi_2)/(2s)$
2	$(1 + \cos\varphi_1\cos\varphi_2 + \cos\varphi_2\sin\varphi_1 - \cos\varphi_1\sin\varphi_2 + \sin\varphi_1\sin\varphi_2)/s$
3	$(1 + \cos\varphi_1\cos\varphi_2 - \cos\varphi_2\sin\varphi_1 + \cos\varphi_1\sin\varphi_2 + \sin\varphi_1\sin\varphi_2)/s$
4	$(1 - \cos\varphi_1 + \cos\varphi_2 - \cos\varphi_1\cos\varphi_2 - \sin\varphi_1 - \cos\varphi_2\sin\varphi_1 + \sin\varphi_2$ $+ \cos\varphi_1\sin\varphi_2 - \sin\varphi_1\sin\varphi_2)/(2s)$
5	$(1 + \cos\varphi_1 - \cos\varphi_2 - \cos\varphi_1\cos\varphi_2 - \sin\varphi_1 + \cos\varphi_2\sin\varphi_1 + \sin\varphi_2$ $- \cos\varphi_1\sin\varphi_2 - \sin\varphi_1\sin\varphi_2)/(2s)$